Section 16.4

Parametrized Surfaces and Surface Integrals

1 Parametrizing Surfaces

by Joseph Phillip Brennan
Jila Niknejad

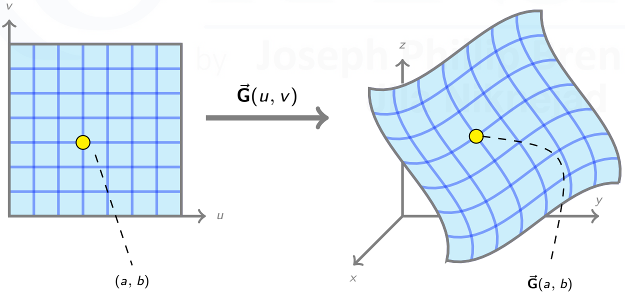
Parametrized Surfaces

A curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a one-dimensional object and can be parametrized by a vector-valued function of one variable:

$$\vec{\mathbf{r}}(t) = \langle x(t),\ y(t) \rangle \quad \text{or} \quad \langle x(t),\ y(t),\ z(t) \rangle\,.$$

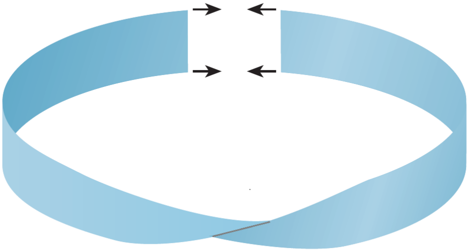
By contrast, a surface in $\mathbb{R}^3$ is a two-dimensional object and can be parametrized by a vector-valued function of two variables:

$$\vec{\mathbf{G}}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v) \rangle\,.$$



Parametrized Surfaces: Quick Preview Of Examples

- $\vec{\mathbf{G}}(u, v) = \langle 2 - 2u + v,\ 3 + 4v,\ -7 + 8u - 5v \rangle$: plane
- $\vec{\mathbf{G}}(u, v) = \langle u,\ v,\ f(u, v) \rangle$: graph of scalar function f
- $\vec{\mathbf{G}}(\theta, \phi) = \langle R\cos(\theta)\sin(\phi),\ R\sin(\theta)\sin(\phi),\ R\cos(\phi) \rangle$: sphere
- $\vec{\mathbf{G}}(\theta, \phi) = \langle A\cos(\theta)\sin(\phi),\ B\sin(\theta)\sin(\phi),\ C\cos(\phi) \rangle$: ellipsoid
- $\vec{\mathbf{G}}(r, \theta) = \langle 2\cos(\theta) + r\cos(\theta/2),\ 2\sin(\theta) + r\cos(\theta/2),\ r\sin(\theta/2) \rangle$: Möbius strip!



Parametrizations and Transformations

- In §15.6, we studied **transformations** (in the context of changing variables in multiple integrals).

  A transformation is a function $G : E \to D$ (where E, D are regions in $\mathbb{R}^2$ or $\mathbb{R}^3$) such that

  - G is one-to-one on the interior of E;
  - G has continuous first-order partial derivatives.

- A **parametrization** of a surface $\mathcal{S}$ in $\mathbb{R}^3$ is defined in almost the same way.

  It is a function $G : D \to \mathcal{S}$ such that

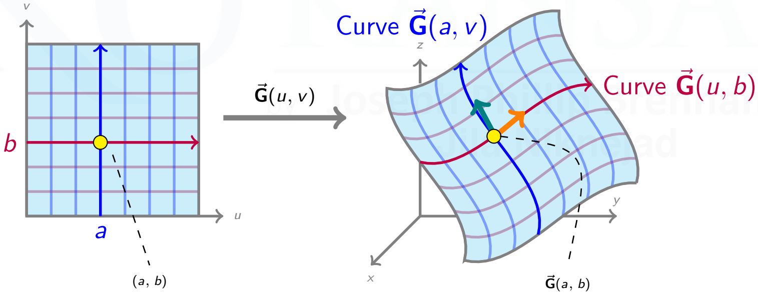
  - G is onto (so that every point in $\mathcal{S}$ is covered;
  - G is one-to-one on the interior of D;
  - G has continuous first-order partial derivatives.

Grid Curves

The **grid curves** of a surface parametrized by $\vec{\mathbf{G}}(u, v)$ are the images of the vertical and horizontal lines in the domain, the uv-plane.

$$\text{vertical line } u = a \quad \xrightarrow{\vec{\mathbf{G}}} \quad \text{grid curve } \vec{\mathbf{G}}(a, v)$$

$$\text{horizontal line } v = b \quad \xrightarrow{\vec{\mathbf{G}}} \quad \text{grid curve } \vec{\mathbf{G}}(u, b)$$



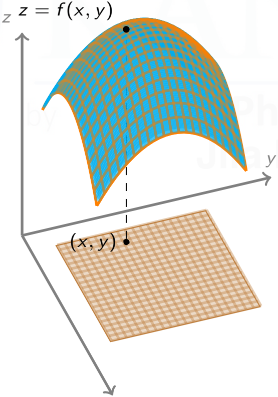
2 Examples of Parametrizations of Common Surfaces

Examples of Common Surfaces, Graphs of Functions

Example 1: If $f(x, y)$ is any function of two variables, its graph is a surface that can be parametrized by

$$\vec{\mathbf{G}}(x, y) = \langle x,\, y,\, f(x, y)\rangle\,.$$

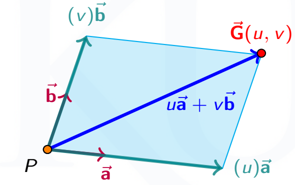
In this case, the grid curves are just the "lifts" of horizontal and vertical lines in $\mathbb{R}^2$ up to the graph.



Examples of Common Surfaces, a Plane

Example 2: The **plane** containing the point P and the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ can be parametrized by

$$\vec{\mathbf{G}}(u,v) = \vec{OP} + u\vec{\mathbf{a}} + v\vec{\mathbf{b}}.$$

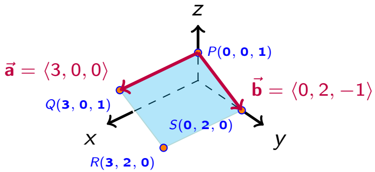


Idea: Every point in the plane can be obtained by starting at P and moving parallel to the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

E.g., a **parametrization** of parallelogram with vertices $P(0,0,1)$, $Q(3,0,1)$, $R(3,2,0)$ and $S(0,2,0)$ is

$$\vec{\mathbf{G}}(u,v) = \underbrace{\langle 0,0,1\rangle}_{P} + u\underbrace{\langle 3,0,0\rangle}_{\vec{PQ}} + v\underbrace{\langle 0,2,-1\rangle}_{\vec{PS}},$$

where $0 \le u, v \le 1$



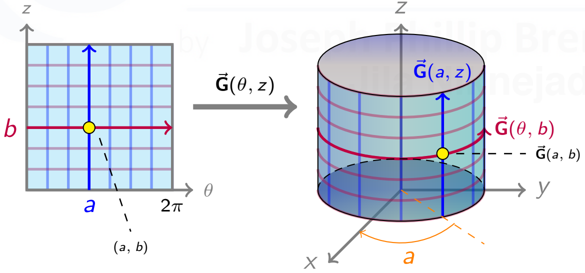
An **equation** of the plane is $3(y-0) + 6(z-1) = 0$.

Examples of Common Surfaces, a Cylinder

Example 3: Find a parametrization for the **cylinder** $x^2 + y^2 = c^2$.

Solution: In cylindrical coordinates, the equation is $\boldsymbol{r = c}$. Thus the transformation from cylindrical to rectangular is a parametrization, with parameters θ and z:

$$\vec{\mathbf{G}}(\theta, z) = \underbrace{\langle c\cos(\theta),\ c\sin(\theta),\ z\rangle}_{\text{Cylindrical Transform. with } r=c}, \qquad \theta \in [0, 2\pi],\quad z \in (-\infty, \infty)$$

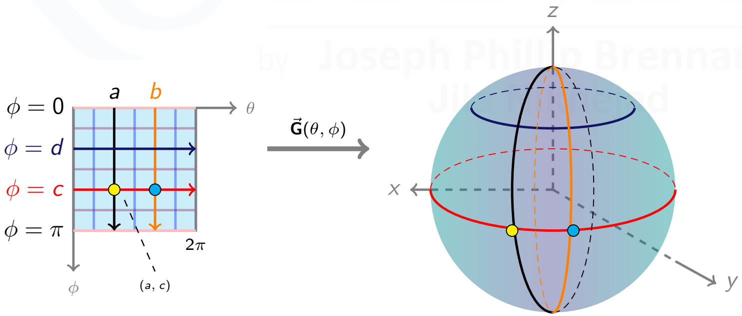


Examples of Common Surfaces, a Sphere

Example 4: Find a parametrization for the **sphere** $x^2 + y^2 + z^2 = c^2$.

Solution: In spherical coordinates, the equation is $\boldsymbol{\rho = c}$. Thus the transformation from spherical to rectangular is a parametrization, with parameters θ and ϕ:

$$\vec{\mathbf{G}}(\theta,\phi) = \underbrace{\langle c\cos(\theta)\sin(\phi),\ c\sin(\theta)\sin(\phi),\ c\cos(\phi)\rangle}_{\text{Spherical Transform. with } \rho=c}, \quad \theta \in [0, 2\pi],\ \phi \in [0, \pi]$$



Examples of Common Surfaces, a Cone

Example 5: Identify the surface parametrized by

$$\vec{\mathbf{G}}(u,v) = \langle u\cos(v),\ u\sin(v),\ u\rangle\ .$$

Solution: Eliminate u and v to obtain an equation in (x, y, z):

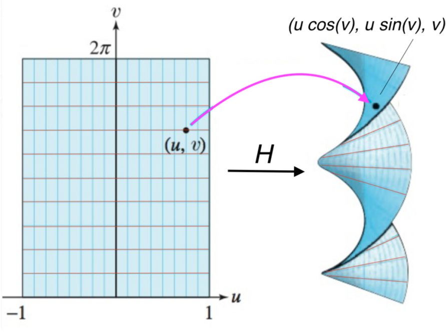
$$x^2 + y^2 = \big(u\cos(v)\big)^2 + \big(u\sin(v)\big)^2 = u^2 = z^2.$$

The parametric surface is a **cone**.

Example 6: The surface parameterized by

$$H(u,v) = \langle u\cos(v), u\sin(v), v\rangle$$

is called a **helicoid.**



Examples of Common Surfaces, a Hyperboloid

Example 7: Find a parametrization for the hyperboloid of one sheet

$$x^2 + y^2 - z^2 = 1$$

Solution: Don't forget that there are an infinite number of parametrizations for any surface!

We will find two methods for parametrizing this surface:

Method I: Set $z = u$. Then $x^2 + y^2 = 1 + u^2$. We can then let

$$x = \sqrt{1+u^2}\cos(v) \qquad y = \sqrt{1+u^2}\sin(v)$$

$$\vec{\mathbf{G}}(u,v) = \left\langle \sqrt{1+u^2}\cos(v), \sqrt{1+u^2}\sin(v), u \right\rangle$$

Method II: Set $z = \tan(u)$. Then $x^2 + y^2 = \sec^2(u)$. We can then let

$$x = \sec(u)\cos(v) \qquad y = \sec(u)\sin(v)$$

$$\vec{\mathbf{G}}(u,v) = \langle \sec(u)\cos(v), \sec(u)\sin(v), \tan(u) \rangle$$

3 Surface Area

by Joseph Phillip Brennan
Jila Niknejad

Surface Area of a Parametrized Surface

In §14.3 we calculated the **arclength** of a parametrized curve and then we used the resulting formula to define line integrals.

In a similar fashion, we will calculate the **surface area** of a parametrized surface and will eventually use it to define **surface integrals**.

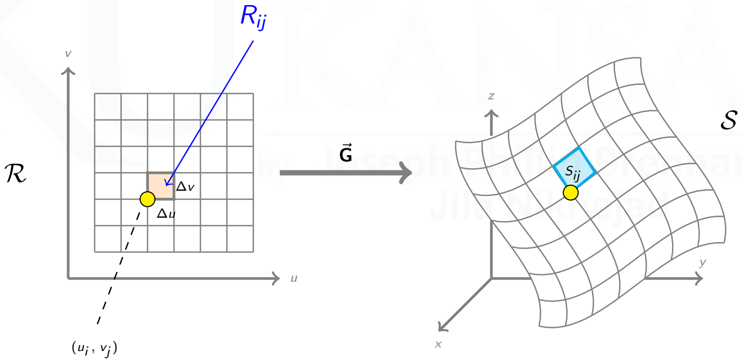
Setup: Suppose we have a parametrization $G : \mathcal{R} \subseteq \mathbb{R}^2 \to \mathcal{S}$, where $\vec{\mathbf{G}}(u, v) = (x(u, v),\ y(u, v),\ z(u, v))$. Define

$$\vec{\mathbf{G}}_u(u, v) = \langle x_u(u, v),\ y_u(u, v),\ z_u(u, v)\rangle\,,$$
$$\vec{\mathbf{G}}_v(u, v) = \langle x_v(u, v),\ y_v(u, v),\ z_v(u, v)\rangle\,.$$

These vectors are tangent to the grid curves at the point $\vec{\mathbf{G}}(u, v)$ on $\mathcal{S}$.

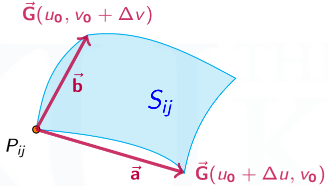
Surface Area of a Parametrized Surface

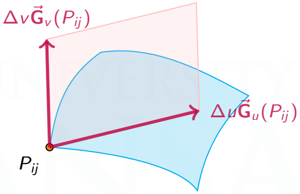
- Partition $\mathcal{R}$ into rectangular subregions R_{ij}.
- Each subregion maps to a piece S_{ij} of the surface $\mathcal{S}$.
- The boundary of S_{ij} is made up of grid curves.



Surface Area of a Parametrized Surface

Similar to the Jacobian calculations of section 15.6, we approximate S_{ij} as a rectangle of width $\vec{\mathbf{G}}_u(u_i, v_j)\,\Delta u$ and length $\vec{\mathbf{G}}_v(u_i, v_j)\,\Delta v$.





Using the parallelogram property of the cross product:

$$\begin{aligned}\text{Area}(S_{ij}) &\approx \|\vec{\mathbf{G}}_u(u_i, v_j)\,\Delta u \times \vec{\mathbf{G}}_v(u_i, v_j)\,\Delta v\| \\ &= \|\vec{\mathbf{G}}_u(u_i, v_j) \times \vec{\mathbf{G}}_v(u_i, v_j)\|\,\Delta u\,\Delta v\end{aligned}$$

Summing the areas into a Riemann sum and taking a limit yields

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\vec{\mathbf{G}}_u(u, v) \times \vec{\mathbf{G}}_v(u, v)\|\, dA$$

Surface Area Formula — Parametrization Version

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\vec{\mathbf{G}}_u(u,v) \times \vec{\mathbf{G}}_v(u,v)\| \, dA$$

Example 8: Find the surface area of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 5$.

Solution: First, parametrize the cylinder and calculate partial derivatives:

$$\begin{aligned} \vec{\mathbf{G}}(u,v) &= \langle 2\cos(u), 2\sin(u), v \rangle \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 5 \\ \vec{\mathbf{G}}_u(u,v) &= \langle -2\sin(u), 2\cos(u), 0 \rangle \\ \vec{\mathbf{G}}_v(u,v) &= \langle 0, 0, 1 \rangle \end{aligned}$$

Then apply the area formula:

$$\begin{aligned} \text{Area}(S) &= \int_0^{2\pi} \int_0^5 \big\| \langle 2\cos(u), 2\sin(u), 0 \rangle \big\| \, dv \, du \\ &= \int_0^{2\pi} \int_0^5 \underbrace{2}_{\text{Radius!}} \, dv \, du = 20\pi \end{aligned}$$

Example 9: Calculate the surface area of the sphere of radius r.

Solution: Recall that the sphere is parametrized by

$$\vec{\mathbf{G}}(\theta, \phi) = \langle r\cos(\theta)\sin(\phi),\ r\sin(\theta)\sin(\phi),\ r\cos(\phi)\rangle$$

for $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$. Therefore

$$\begin{aligned}
\vec{\mathbf{G}}_\theta &= \langle -r\sin(\theta)\sin(\phi),\ r\cos(\theta)\sin(\phi),\ 0\rangle \\
\vec{\mathbf{G}}_\phi &= \langle r\cos(\theta)\cos(\phi),\ r\sin(\theta)\cos(\phi),\ -r\sin(\phi)\rangle \\
\vec{\mathbf{G}}_\theta \times \vec{\mathbf{G}}_\phi &= -r^2\sin(\phi)\ \langle \cos(\theta)\sin(\phi),\ \sin(\theta)\sin(\phi),\ \cos(\phi)\rangle \\
\|\vec{\mathbf{G}}_\theta \times \vec{\mathbf{G}}_\phi\| &= r^2\sin(\phi)
\end{aligned}$$

(That should look familiar!) So the surface area of the sphere is

$$\int_0^\pi \int_0^{2\pi} \underbrace{r^2\sin(\phi)}_{\text{Plug in } \rho = r \text{ in Jacobian!}} \, d\theta\, d\phi = 2\pi r^2 \int_0^\pi \sin(\phi)\, d\phi = 4\pi r^2.$$

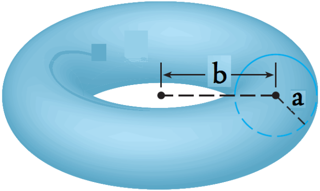
Torus (Optional)

Example 10: A torus (surface of a donut) can be obtained by rotating the circle on the yz-plane centered at $(0, b, 0)$ with radius a $(b > a > 0)$ about the z-axis. The torus has a parametrization $\vec{\mathbf{G}}(\theta, \alpha)$ given by

$$x(\theta, \alpha) = b\cos(\theta) + a\cos(\alpha)\cos(\theta)$$
$$y(\theta, \alpha) = b\sin(\theta) + a\cos(\alpha)\sin(\theta)$$
$$z(\theta, \alpha) = a\sin(\alpha)$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \alpha \leq 2\pi$.

Find the surface area of the torus.



Solution: First, calculate $\|\vec{\mathbf{G}}_\theta \times \vec{\mathbf{G}}_\alpha\| = a(b + a\cos(\alpha))$.

Then, use the surface area formula:

$$\int_0^{2\pi}\int_0^{2\pi} a(b + a\cos(\alpha))\,d\theta\,d\alpha = 2\pi a\int_0^{2\pi}(b + a\cos(\alpha))\,d\alpha$$
$$= 4\pi^2 ab.$$

Surface Area of a Graph

Recall that the **graph** of $z = f(x, y)$ has a natural parametrization

$$\vec{\mathbf{G}}(x, y) = \langle x, y, f(x, y) \rangle.$$

The partial derivative vectors are

$$\vec{\mathbf{G}}_x(x, y) = \langle 1, 0, f_x(x, y) \rangle \qquad \vec{\mathbf{G}}_y(x, y) = \langle 0, 1, f_y(x, y) \rangle$$

and

$$\vec{\mathbf{G}}_x \times \vec{\mathbf{G}}_y = \langle -f_x, -f_y, 1 \rangle \qquad \therefore \qquad \|\vec{\mathbf{G}}_x \times \vec{\mathbf{G}}_y\| = \sqrt{1 + f_x^2 + f_y^2}.$$

Surface Area Formula — Graph Version

The part of the graph of $z = f(x, y)$ over a region $\mathcal{D}$ in $\mathbb{R}^2$ has surface area

$$\iint_{\mathcal{D}} \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA \;=\; \iint_{\mathcal{D}} \sqrt{1 + z_x^2 + z_y^2}\, dA.$$

Example 11: Calculate the surface area of the part of the saddle surface $z = xy$ lying over the unit disk $\mathcal{D}$.

Solution: The surface area formula (Graph Version) gives

$$\text{area} = \iint_{\mathcal{D}} \underbrace{\sqrt{1 + y^2 + x^2}}_{\sqrt{z_x^2 + z_y^2 + 1}} \, dA.$$

In polar coordinates, the integral becomes

$$\int_0^{2\pi} \int_0^1 \sqrt{1 + r^2} \underbrace{r}_{jac} \, dr \, d\theta = 2\pi \int_0^1 r\sqrt{1 + r^2} \, dr.$$

Substitute $u = 1 + r^2$, $du = 2r\, dr$ to get

$$\frac{2\pi}{2} \int_1^2 \sqrt{u} \, du = \pi \frac{2}{3} u^{3/2} \Big|_1^2 = \frac{2\pi}{3}(2\sqrt{2} - 1) \approx 3.829.$$

4 Scalar Surface Integrals

Scalar Surface Integrals

Suppose that $f(x, y, z)$ is a function on a surface $\mathcal{S}$ parametrized by $\vec{\mathbf{G}}(u, v)$ over the domain $\mathcal{R}$.

The **surface integral** of f over $\mathcal{S}$ is defined as

$$\iint_{\mathcal{S}} f(x, y, z)\, dS = \iint_{\mathcal{R}} f\left(\vec{\mathbf{G}}(u, v)\right) \|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|\, dA$$

- If $f(x, y, z) = 1$ then the surface integral is just surface area.
- If $f(x, y, z)$ represents the density of stuff per unit area, then the surface integral is the total amount of stuff.
- The symbol dS is called the **surface element** or the **area element**:

$$dS = \|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|\, dA_{uv}.$$

Scalar Surface Integrals: Examples

Example 12: Evaluate $\iint_{\mathcal{S}} x^2 \, dS$, where $\mathcal{S}$ is $x^2 + y^2 + z^2 = 9$.

Solution: First, parametrize the sphere using spherical coordinates.

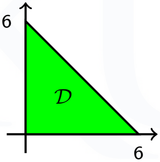
$$\vec{\mathbf{G}}(\theta,\phi) = \underbrace{\langle 3\sin(\phi)\cos(\theta),\ 3\sin(\phi)\sin(\theta),\ 3\cos(\phi)\rangle}_{\text{Spherical Tranform. with } \rho=3} \qquad \|\vec{\mathbf{G}}_\theta \times \vec{\mathbf{G}}_\phi\| = \underbrace{9\sin(\phi)}_{\rho^2 \sin(\phi)}$$

$$\begin{aligned}
\iint_{\mathcal{S}} x^2 \, dS &= \int_0^{2\pi}\int_0^{\pi} 81\,(\sin(\phi)\cos(\theta))^2 \sin(\phi)\, d\phi\, d\theta \\
&= 81\left(\int_0^{2\pi} \underbrace{\cos^2(\theta)}_{\frac{1+\cos(2\theta)}{2}} d\theta\right)\left(\int_0^{\pi} \underbrace{\sin^3(\phi)}_{\sin(\phi)(1-\cos^2(\phi))} d\phi\right) \\
&= 81\left(\frac{\cos^3(\phi) - 3\cos(\phi)}{3}\bigg|_0^{\pi}\right)\left(\frac{2\theta + \sin(2\theta)}{4}\bigg|_0^{2\pi}\right) = 108\pi.
\end{aligned}$$

Scalar Surface Integrals: Examples

Example 13: Evaluate $\iint_{\mathcal{S}} yz\, dS$, where $\mathcal{S}$ is the part of the plane $x + y + z = 6$ that lies in the first octant.

<u>Solution:</u> Parametrize $\mathcal{S}$ as $\vec{\mathbf{G}}(x, y) = (x, y, 6 - x - y)$, where $(x, y) \in \mathcal{D}$.



$$\vec{\mathbf{G}}_x = \langle 1, 0, -1 \rangle \qquad \vec{\mathbf{G}}_x \times \vec{\mathbf{G}}_y = \langle 1, 1, 1 \rangle$$

$$\vec{\mathbf{G}}_y = \langle 0, 1, -1 \rangle \qquad \|\vec{\mathbf{G}}_x \times \vec{\mathbf{G}}_y\| = \sqrt{3}$$

$$\iint_{\mathcal{S}} yz\, dS = \iint_{\mathcal{D}} y(6 - x - y)\sqrt{3}\, dA = \sqrt{3} \int_0^6 \int_0^{6-y} 6y - xy - y^2\, dx\, dy$$

$$= \sqrt{3} \int_0^6 \frac{y^3}{2} - 6y^2 + 18y\, dy = 54\sqrt{3}$$